APPLICATIONS OF NON-ARCHIMEDEAN INTEGRATION TO THE L-SERIES OF τ-SHEAVES

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ABSTRACT. Let $\underline{\mathcal{F}}$ be a τ-sheaf. Building on previous work of Drinfeld, Anderson, Taguchi, and Wan, Böckle and Pink [BP1] develop a cohomology theory for $\underline{\mathcal{F}}$. In [Boc1] Böckle uses this theory to establish the analytic continuation of the L-series associated to $\underline{\mathcal{F}}$ (which is a characteristic p valued "Dirichlet series") *and* the logarithmic growth of the degrees of its special polynomials. In this paper we shall show that this logarithmic growth is all that is needed to analytically continue the original L-series as well as *all* associated partial L-series. Moreover, we show that the degrees of the special polynomials attached to the partial L-series also grow logarithmically. Our tools are Böckle's original results, non-Archimedean integration, and the very strong estimates of Y. Amice [Am1]. Along the way, we define certain natural modules associated with non-Archimedean measures (in the characteristic 0 case as well as in characteristic p).

1. INTRODUCTION

In his original work [R1] on his zeta function, Riemann established that the density of zeroes up to level T in the critical strip is approximately $\frac{1}{2\pi}\log(\frac{T}{2\pi})$. Since then similar results have been established for general L-series.

In the arithmetic of function fields over finite fields, logarithmic growth manifests itself for characteristic p valued zeta functions in terms of the degrees of their associated "special polynomials" (see Subsection 3.2.3). This was first noted by the author in the explicit measure calculations of Dinesh Thakur [Th1] for $\mathbb{F}_r[T]$.

More generally, let k be an arbitrary global function field with full field of constants $\mathbb{F}_r$ and let ∞ be a fixed place. Set A to be the Dedekind domain of elements of k which are integral at all places outside ∞. Let k_1 be a finite extension of k and let ϕ be a Drinfeld module over k_1. As in §8.6 of [Go4], one can define the L-series $L(s)$ of ϕ which is a "Dirichlet series" $\sum_I c_I I^{-s}$, I an ideal of A, in finite characteristic. Using elementary estimates, (Lemma 8.8.1 of [Go4]) it was shown when ϕ has rank 1 (or complex multiplication etc.) that $L(s)$ has an analytic continuation to an entire function. Moreover analytic continuations can then also be established for the interpolations associated to $L(s)$ at finite primes in the sense of Subsection 3.2.3.

The estimates of [Go4] allow one to establish the existence of special polynomials in the general rank 1 case but give poor estimates on the degrees of these polynomials. As mentioned, the explicit calculation of Thakur in [Th1], as well as the calculations of Newton polygons by Wan, Diaz-Vargas and Sheats ([Wa1], [DV1] and [Sh1]), show that these degrees in fact grow logarithmically and that this logarithmic growth reflect rationality (in terms of the complete field k_∞) of the zeroes of such function. Moreover, this logarithmic growth,

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This work is dedicated to the memory of Arnold Ross; a wonderful colleague who had an enormous positive influence on the mathematics community.

when combined with the deep a-priori estimates of Amice [Am1], actually provides the analytic continuation of these L-series at *all* places of k. It thus became reasonable to also expect such logarithmic growth for the degrees of special polynomials associated to L-series of general rank Drinfeld modules, t-modules etc. And, indeed, such a basic result was recently established by G. Böckle in [Boc1] as a stepping stone in his analytic continuation of such L-series.

The approach to characteristic p L-series in [Boc1] is via cohomology. Due to the labors of Drinfeld, G. Anderson, Y. Taguchi, D. Wan, R. Pink and Böckle, the basic construction of characteristic p arithmetic has evolved from Drinfeld modules to "τ-sheaves" which are simply coherent sheaves over the product of a base scheme X with $\mathrm{Spec}(A)$ equipped with a Frobenius-linear morphism τ (see Definition 12). In [BP1], Pink and Böckle show how to embed the τ-sheaves into a category of "crystals" which possesses a good cohomology theory and Lefschetz fixed point theorem. It is this cohomology theory that Böckle uses to establish the logarithmic growth of the degrees of the special polynomials (and the analytic continuation of these functions at ∞ and *all* interpolations at finite primes) in very great generality.

Let $L(s)$ be the L-series of a τ-sheaf of the type shown to be entire in [Boc1] (see, e.g., Theorem 2 below). In this paper we show how the logarithmic growth of the degrees of $L(x-j)$ is enough to establish the analytic continuation and logarithmic growth of *any partial L-series* (see Definition 25) associated to $L(s)$. We say that such a Dirichlet series is "in the motivic class $\mathcal{M}$" as it is our expectation that the only way to provide non-trivial examples is precisely via τ-sheaves. The idea behind the proof is again to express $L(s)$ as a non-Archimedean integral which a priori is valid in the "half-plane" where $L(s)$ has absolute convergence (in the sense of Remark 8). Then one uses the logarithmic growth of the degrees of $L(x,-j)$ to show that the measures so obtained blow up very slowly. Combined with Amice's estimates, the analytic continuation follows. Then playing off Amice's estimates against certain a priori estimates on coefficients allows one to also obtain the logarithmic growth for the degrees of the special polynomials associated to the partial L-series.

A very interesting feature of the proof of the logarithmic growth of the degrees of special polynomials associated to partial L-series is the way that the theories at the place ∞ and *all* finite places intertwine. Indeed, to establish the logarithmic growth of a partial L-series defined modulo a given place w we need crucially to use the w-adic theory associated to $L(s)$.

Along the way, we elucidate some of the formalism associated with non-Archimedean integration both in finite characteristic and characteristic 0. In particular we show how the convolution product of measures comes equipped with certain canonical associated modules. In finite characteristic, these modules give a concrete realization of the space of measures as "differential operators" which was previously only known abstractly (e.g., [Go8]).

Our results, along with those of Böckle, Pink, Taguchi, and Wan, make it very reasonable to hope that a deeper theory of the zeroes will eventually be found. Indeed, the results of [Wa1], [DV1], [Sh1] give far more information in certain special cases than is obtainable from the estimates given here. As of now this theory would seem to involve first a deeper understanding of the relationship between the characteristic p L-series and modular forms associated to Drinfeld modules as established in [Boc2] (and presented in [Go7]). Indeed, Böckle in [Boc2] associates a τ-sheaf to a cusp-form via Hecke operators; thus cusp-forms *also* give rise to characteristic p valued L-series.

The reader may wonder why one could not approach our results by simply using twists of the L-series by abelian characters and then solving for the partial L-series. However, there are simply not enough characters with values in finite characteristic for this to work in general.

In this paper we have worked with completely arbitrary A. All the results go through in this case, but there are a number of associated technicalities that must be dealt with. These technicalities involve making sense of "I^s" when I is not generated by a "monic element" (in the sense of Subsection 3.2.1 which generalizes the usual notion of monic polynomial). Of course when $A = \mathbb{F}_r[T]$, all I obviously *are* so generated, and thus the technicalities vanish. Therefore the reader is well advised to first read this paper with $A = \mathbb{F}_r[T]$.

It is my pleasure to thank Gebhard Böckle for his help in understanding the results in [BP1], [Boc1], and [Boc2] and for his comments on early versions of this work. These comments greatly helped clarify the proof of our main result. Similarly, I am also indebted to Keith Conrad and the referee for helpful comments. It is finally my pleasure to thank Zifeng Yang who pointed out that my original proof of the logarithmic growth (of the degrees of the special polynomials) at the infinite place would also work at the finite places. Indeed, because the degree of a principal divisor on a complete curve must equal 0, the order of zero of an element of A at a prime of A is obviously controlled by the order of its pole at ∞.

2. Review of non-Archimedean integration

2.1. **General theory.** In this section K will be a non-Archimedean local field, of any characteristic, with maximal compact subring R_K and associated maximal ideal M_K. Thus $\mathbb{F}_K := R_K/M_K$ is a finite field and we denote its order by q_K. Let $|?| = |?|_K$ be the absolute value on K defined by $|\pi| = q_K^{-1}$, where $M_K = (\pi)$ (so that $R_K = \{x \in K \mid |x| \leq 1\}$ and $M_K = \{x \in K \mid |x| < 1\}$). We let $v(?) = v_K(?)$ be the additive valuation associated to $|?|$ with $v(\pi) = 1$. Let $\bar{K}$ be a fixed algebraic closure of K equipped with the canonical extension of $|?|$ and $v(?)$. Finally let $K^{\text{sep}} \subseteq \bar{K}$ be the separable closure.

Let $L \subset \bar{K}$ be a finite extension of K with integers R_L (so that L is still a local field).

Definition 1. An *R_L-valued measure* on R_K is a finitely additive R_L-valued function on the compact open subsets of R_K.

More generally, one defines an *L-valued measure on R_K* to be a finitely additive L-valued function μ on the compact open subsets of R_K with bounded image in L. One sees immediately that the μ is an L-valued measure if and only if there exists $a \neq 0 \in L$ such that $a\mu$ is an R_L-valued measure. We will denote the space of R_L-valued measures on R_K by $\mathcal{M}(R_K, R_L)$ and the space of L-valued measures by $\mathcal{M}(R_K, L)$; so $\mathcal{M}(R_K, L) = L \otimes \mathcal{M}(R_K, R_L)$.

Remark 1. Arbitrary (i.e., possibly unbounded) L-valued finitely additive functions on the compact opens of R_K are called *L-valued distributions.*

Let $f\colon R_K \to L$ be a continuous function and let $\mu \in \mathcal{M}(R_K, L)$. One defines Riemann sums associated to f and μ in the obvious manner. As R_K is compact, f is also uniformly continuous. Therefore it is easy to see that the Riemann sums converge to an element of L which is naturally denoted $\displaystyle\int_{R_K} f(x)\, d\mu(x)$.

Let E be a vector space over L. A map $\|\ \|\colon E \to \mathbb{R}$ is a *norm* if and only if

(1) $\|x\| = 0 \Leftrightarrow x = 0 \in E$,

(2) $\|x+y\| \leq \max\{\|x\|, \|y\|\}$,
(3) $\|ax\| = |a|\|x\|$ for $a \in L$ and $x \in E$.

The norm $\|?\|$ induces an ultrametric ρ on E by $\rho(x,y) := \|x-y\|$.

Definition 2. A *Banach space over* L is a complete normed L-vector space.

Let E be an L-Banach space. We say that E is *separable* if and only if E contains a dense countable subset. From now on we will only consider separable L-Banach spaces.

Definition 3. Let E be a Banach space and $\{e_i\}_{i=0}^{\infty}$ be a countable subset of E. We say that $\{e_i\}$ is an *orthonormal* basis (or *Banach basis*) for E if and only if

(1) every $x \in E$ can be written uniquely as a convergent sum $x = \sum_{i=0}^{\infty} c_i e_i$ for $\{c_i\} \subset K$, $c_i \to 0$ as $i \to \infty$,
(2) $\|x\| = \sup_i\{c_i\}$.

Example 1. Let $\mathcal{C}(R_K, L)$ be the L-vector space of continuous L-valued functions on R_K equipped with the sup norm $\|f\|$ for continuous f; it is easy to see that $\mathcal{C}(R_K, L)$ is an L-Banach space. A theorem due to Kaplansky ([Ka1] or Th. 5.28 in [vR1]) assures us that the polynomial functions are dense in $\mathcal{C}(R_K, L)$. It follows readily that $\mathcal{C}(R_K, L)$ is separable.

The existence of orthonormal bases $\{Q_n(x)\}$, where $Q_n(x)$ is a polynomial of degree n, for $\mathcal{C}(R_K, L)$ (where L is local as above) will be of critical importance to us. In [Am1] (see also [Ya1]) Y. Amice constructs such a basis by first using Newton interpolation involving certain "very well distributed" sequences of elements of R_K to construct polynomials $\{p_n(x)\}$ of degree n. Then the Banach basis $\{Q_n(x)\}$ for $\mathcal{C}(R_K, L)$ is defined by $Q_n(x) := p_n(x)/s_n$ for all n where $v_K(s_n) = \sum_{i=1}^{\infty}[n/q_K^i]$. In the case where $R_K = \mathbb{F}_r[[T]]$, K. Conrad in [Co1] shows how to use an orthonormal basis of the Banach space $\mathcal{L}_{\mathbb{F}_r}(R_K, L)$ of all $\mathbb{F}_r$-*linear* continuous functions (obviously a closed subspace of $\mathcal{C}(R_K, L)$) to construct a polynomial basis for the space of all continuous functions via the "digit principle." We shall have more to say about this later in Subsection 2.3.1.

Let μ be a measure on R_K. Set

$$\left\{b_n := \int_{R_K} Q_n(x)\, d\mu(x)\,, \quad n = 0, 2\ldots\right\}. \tag{1}$$

We call $\{b_n\}$ the *measure coefficients associated to the basis* $\{Q_n(x)\}$. The boundedness of μ immediately implies that $\{b_n\} \subset L$ is also bounded. Moreover, let $f(x) = \sum_{n=0}^{\infty} a_n Q_n(x)$ be a continuous function, where the *expansion coefficients* $\{a_n\}$ lie in L and $a_n \to 0$ as $n \to \infty$. Continuity implies that

$$\int_{R_K} f(x)\, d\mu(x) = \sum_{n=0}^{\infty} a_n b_n\,. \tag{2}$$

Note that the locally constant functions are also continuous; thus *any* bounded sequence $\{b_n\} \subset L$ gives rise to a bounded measure by Equation 2. Consequently a given choice of orthonormal basis for $\mathcal{C}(R_K, L)$ immediately gives a corresponding isomorphism of the space of measures with the space of bounded sequences with coefficients in L.

Definition 4. Let $\alpha \in R_K$ and $f \in \mathcal{C}(R_K, L)$. We define the *Dirac measure* δ_α associated to α by

$$\int_{R_K} f(x)\, d\delta_\alpha(x) := f(\alpha)\,.$$

Using the ideas just presented, one sees readily that the Dirac measure is indeed a bounded measure in the sense of Definition 1. The Dirac measures provide the basic building blocks for the constructions given in this paper.

The space of measures is also an L-algebra via the convolution product in the standard manner which we recall in the next definition.

Definition 5. Let μ and ν be elements of $\mathcal{M}(R_K, L)$. Let $f\colon R_K \to L$ be continuous. We define the *convolution* $\mu * \nu \in \mathcal{M}(R_K, L)$ by

$$\int_{R_K} f(u)\, d\mu * \nu(u) := \int_{R_K} \int_{R_K} f(x+y)\, d\mu(y) d\nu(x)\,. \tag{3}$$

It is easy to see that Equation 3 does indeed define a new measure and makes $\mathcal{M}(R_K, L)$ into a commutative L-algebra.

Let α and β be two elements of R_K. By definition one has

$$\delta_\alpha * \delta_\beta = \delta_{\alpha+\beta}\,. \tag{4}$$

2.2. **The characterization of locally analytic functions.** In this subsection we will review the basic results of Y. Amice [Am1] (see also [Ya1]) that permit us to characterize those $f \in \mathcal{C}(R_K, L)$ which are locally analytic.

Let $0 \neq \rho \in R_K$ with $t = |\rho|$. Let α be another element of R_K. The closed ball $B_{\alpha,t}$ around α of radius t is defined, as usual, by

$$B_{\alpha,t} := \{x \in R_K \mid |x - \alpha| \le t\}\,.$$

Definition 6. A continuous function $f\colon B_{\alpha,t} \to L$ is *analytic on* $B_{\alpha,t}$ if and only if f may be expressed as

$$f(x) = \sum_{n=0}^{\infty} b_n \left(\frac{x-\alpha}{\rho}\right)^n, \tag{5}$$

where $\{b_n\} \subset L$ and $b_n \to 0$ as $n \to \infty$.

The norm $\|f\|_B$ of f on $B := B_{\alpha,t}$ is defined by

$$\|f\|_B := \sup_{n=0}^{\infty}\{|b_n|\}\,. \tag{6}$$

Clearly the set of functions analytic on B forms an algebra which is topologically isomorphic to the Tate algebra $L\langle\langle u\rangle\rangle$ of power series $\sum_{i=0}^{\infty} c_i u^i$ converging on the closed unit disc (i.e., $c_i \to 0$ as $i \to \infty$). Let $\bar{R}_K \subset \bar{K}$ be the ring of integers. Standard results on Tate algebras then imply the basic result

$$\|f\|_B = \sup_{\lambda \in \alpha + \rho \bar{R}_K} \{|f(\lambda)|\}\,. \tag{7}$$

Definition 7. A continuous function $f \in \mathcal{C}(R_K, L)$ is said to be *locally analytic* if for each $\alpha \in R_K$ there exists $t_\alpha > 0$ such that f is analytic on B_{α,t_α}.

Now let $\pi \in R_K$ be a uniformizing parameter.

Definition 8. We say that a locally analytic function f has *order* h, where h is a non-negative integer, if we can choose $t_\alpha \ge |\pi|^h$ for all $\alpha \in R_K$.

Definition 8 is equivalent to requiring f to be analytic on each coset of M_K^h. Note that by compactness one can find a finite number of α with $\{B_{\alpha,t_\alpha}\}$ covering R_K. Thus every locally analytic function has order h for some non-negative h.

We denote the space of locally analytic L-valued functions on R_K by $\mathcal{L}A(R_K, L)$ and those of order h by $\mathcal{L}A_h(R_K, L)$. Clearly $\mathcal{L}A(R_K, L) = \bigcup \mathcal{L}A_h(R_K, L)$.

Definition 9. Let $R_K = \bigcup_{j=0}^m B_j$ where the balls $B_j := B_{\alpha_j, |\pi|^h}$ are mutually disjoint. Let $f \in \mathcal{L}A_h(R_K, L)$. Then we set

$$\|f\|_h := \max_j \{\|f\|_{B_j}\}\,. \tag{8}$$

One checks easily that Definition 9 makes the space $\mathcal{L}A_h(R_K, L)$ a Banach space. One also readily sees that a sequence of functions $\{f_i\}$ converging to a function f in $\mathcal{L}A_h(R_K, L)$ will also converge to f in $\mathcal{L}A_{h'}(R_K, L)$ for any $h' \geq h$. By Equation 7, $\{f_i\}$ also converges to f in the sup norm on continuous functions with domain R_K.

Let $R_K = \bigcup B_j$ as in Definition 9. Let $\{\chi_{j,n}(x)\}$, $j = 0, \cdots, m$ and $n \geq 0$, be the set of locally analytic functions defined by

$$\chi_{j,n}(x) := \begin{cases} \left(\frac{x-\alpha_j}{\pi^h}\right)^n & \text{for } x \in B_j \\ 0 & \text{otherwise}\,. \end{cases} \tag{9}$$

It is very simple to see that $\{\chi_{j,n}(x)\}$ is an orthonormal basis for $\mathcal{L}A_h(R_K, L)$; thus $\mathcal{L}A_h(R_K, L)$ is also a separable Banach space.

Let $\{Q_n(x) = p_n(x)/s_n\}$ be the orthonormal basis for $\mathcal{C}(R_K, L)$ constructed by Amice as mentioned above, and let $f \in \mathcal{C}(R_K, L)$ be expressed as $f(x) = \sum_{n=0}^\infty a_n Q_n(x)$ where $\{a_n\} \subset L$ and $a_n \to 0$ as $n \to \infty$. Put

$$\gamma = \gamma_f := \liminf_n \{v(a_n)/n\}\,.$$

We then have the following results [Am1] (see also [Ya1]).

Theorem 1. 1. *The set $\{p_n(x)\}$ forms an orthonormal basis for the Tate algebra of locally analytic functions of order* 0 *(i.e., functions analytic on the closed unit disc). Moreover, for $h \geq 1$ the collection $\{p_n(x)/s_{n,h}\}$, $\{s_{n,h}\} \subset L$, forms an orthonormal basis for $\mathcal{L}A_h(R_K, L)$ if and only if*

$$v_K(s_{n,h}) = \sum_{i=1}^h [n/q_K^i]\,, \tag{10}$$

(where [?] *is the standard greatest integer function).*
2. *The function f is locally analytic of order h if and only if $v(a_n) - \sum_{i=h+1}^\infty \left[\frac{n}{q^i}\right]$ tends to ∞ as $n \to \infty$.*
3. *The function f is locally analytic if and only if $\gamma > 0$. In this case, set*

$$l := \max\{0, 1 + [-\log(\gamma(q-1))/\log q]\}\,.$$

Then f is locally analytic of order $h \geq l$.

Note that Part 1 of Theorem 1 can easily be restated in terms of $\{Q_n(x)\}$. Using this reformulation, in [Ya1], Z. Yang shows that Theorem 1 remains true for *any* polynomial orthonormal basis $\{h_n(x)\}$ for $\mathcal{C}(R_K, L)$ with $\deg h_n(x) = n$ all n. In particular, it holds true for the Conrad bases $\{G_{E,n}(x)\}$ mentioned above (see also Equation 14).

Example 2. Let $K = \mathbb{Q}_p$, $R_K = \mathbb{Z}_p$, $|?| = |?|_p$, etc., and let $\{\binom{x}{i}\}$ be the standard Mahler basis for $\mathcal{C}(\mathbb{Z}_p, L)$. By definition, $i!\binom{x}{i} \in \mathbb{Z}[x]$ and is monic of degree i; thus $\{i!\binom{x}{i}\}$ is an orthonormal basis for the Tate algebra $L\langle\langle x\rangle\rangle$ of functions regular on the closed unit disc; i.e., locally analytic functions on $\mathbb{Z}_p$ of order 0. In particular such a function f can then be written

$$f(x) = \sum_{i\geq 0} c_i i! \binom{x}{i}, \tag{11}$$

where $c_i \to 0$ as $i \to \infty$. By Equation 7 we see that if $\{a_i\}$ are the Mahler coefficients of f then $a_i = i!c_i$. Standard results on the p-adic valuation of $i!$ then give a simple proof of Part 2 of Theorem 1 in the case $h = 0$. The general proof in [Am1] is given along similar lines.

Remark 2. Let $f(x) = \sum a_n Q_n(x)$ be locally-analytic of some order $h \geq 0$ as above. Then part 1 of Theorem 1, and the above example, make it clear that the estimates on $v(a_n)$ depend *only* on $\|f\|_h$. Thus if we have a family $\{f_i(x) = \sum a_{n,i} Q_n(x)\}$ of locally analytic functions of fixed order h *and* with constant (or bounded) norm, then the estimates we obtain on $v(a_{n,i})$ are independent of i. This observation is essential for our main result, Theorem 3.

Finally, recall that in Equation 2 we expressed the integral of a continuous function f against a measure μ as $\sum a_n b_n$ where $f(x) = \sum a_n Q_n(x)$ and $\{b_n\}$ are the measure coefficients associated to $\{Q_n(x)\}$. The impact of Amice's Theorem is the following. Let f be locally analytic so that $a_n \to 0$ very quickly. Then we may integrate all such functions f against a distribution with coefficients $\{b_n\}$ (defined in the obvious sense) which may *not* be bounded, as long as $\sum a_n b_n$ converges. Such distributions are said to be *tempered* and they can readily be described (see [Am2]). This is the primary technique used in our main result Theorem 3.

2.3. **Appendix: Associated modules.** The results in this appendix elaborate some of the structure associated to the convolution product of measures. In particular, they explain some earlier calculations [Go8] involving measures in the characteristic p theory. They are not, however, used in the proof of our main results.

We shall explain here how the convolution construction on measures also allows one to make $\mathcal{C}(R_K, L)$ into a natural $\mathcal{M}(R_K, L)$-module.

Definition 10. Let $f \in \mathcal{C}(R_K, L)$ and let $\mu \in \mathcal{M}(R_K, L)$. We define $\mu * f \in \mathcal{C}(R_K, L)$ by

$$\mu * f(x) = (\mu * f)(x) := \int_{R_K} f(x+y)\, d\mu(y)\,. \tag{12}$$

Remark 3. We have used the notation $(\mu, f) \mapsto \mu * f$ to distinguish Definition 10, which associates a continuous function in x to (μ, f), from the usual scalar-valued pairing $(\mu, f) \mapsto \int_{R_K} f(x)\, d\mu(x)$. Note also that constructions similar to Definition 10 are well known in classical analysis.

For instance, Definition 10 immediately gives $\delta_\alpha * f(x) = f(x+\alpha)$ for $\alpha \in R_K$ and Dirac measure δ_α. Moreover, for $\alpha \in R_K$ one sees that

$$\mu * f(\alpha) = \int_{R_K} f(x)\, d\delta_\alpha * \mu(x)\,. \tag{13}$$

Conversely, of course, the space of measures, $\mathcal{M}(R_K, L)$, is a natural $\mathcal{C}(R_K, L)$-module where $(f, \mu) \mapsto f(x)\, d\mu(x)$ as usual.

Let μ be a measure as above and $f \in \mathcal{L}A_h(R_K, L)$. Using Amice's result, Theorem 1, one sees that $\mu * f$ also belongs to $\mathcal{L}A_h(R_K, L)$.

2.3.1. *The characteristic p case.* Definition 10 leads to a remarkable differential formalism in the characteristic p case. Indeed, let $R_K = \mathbb{F}_r[[T]]$, $K = \mathbb{F}_r((T))$, and L a finite extension of K. The Conrad bases of [Co1] (in particular, the Carlitz polynomials [Go8]) lead to the formalism of *differential calculus* in the above module action. To see this, let $E = \{e_i\}$ be a Banach basis of the $\mathbb{F}_r$-linear functions $\mathcal{L}_{\mathbb{F}_r}(R_K, L)$. Let n be a non-negative integer written r-adically as $\sum_{t=0}^m b_t r^t$, $0 \le b_t \le r-1$ all t. Following Carlitz one sets

$$G_{E,n}(x) := \prod_{t=0}^{m} e_t(x)^{b_t}\,. \tag{14}$$

Conrad shows that $\{G_{E,n}(x)\}$ is then a Banach basis for $\mathcal{C}(R_K, L)$.

Standard congruences for binomial coefficients, and the linearity of $e_i(x)$ all i, now immediately imply that $\{G_{E,n}(x)\}$ satisfies the binomial theorem; that is

$$G_{E,n}(x+y) = \sum_{i=0}^{n} \binom{n}{i} G_{E,i}(x) G_{E,n-i}(y)\,.$$

As $\{G_{E,n}(x)\}$ is a Banach basis for $\mathcal{C}(R_K, L)$, each continuous f can be written uniquely as $f(x) = \sum_{n=0}^{\infty} b_n G_{E,n}(x)$ where $b_n \to 0$ as $n \to \infty$. Finally we formally set

$$\frac{D_E^i}{i!} = \frac{\frac{d^i}{dE^i}}{i!}$$

to be the measure determined by

$$\int_{\mathbb{F}_r[[T]]} f(x)\, d\frac{D_E^i}{i!}(x) := b_i\,. \tag{15}$$

Then Equation 12 immediately implies

$$\frac{D_E^i}{i!} * G_{E,n}(x) = \binom{n}{i} G_{E,n-i}(x)\,. \tag{16}$$

The analogy with the usual divided-derivatives on power-series is now obvious.

The reader can easily check that, as operators on $\mathcal{C}(R_K, L)$, one has $\dfrac{D_E^i}{i!}\dfrac{D_E^j}{j!} = \dfrac{D_E^{j+i}}{(j+i)!}$. This establishes again that the convolution algebra of L-valued measures on $\mathbb{F}_r[[T]]$ is isomorphic to the algebra of formal divided derivatives [Go8] (which is itself isomorphic to the algebra of formal divided power series). Moreover, if E' is another Banach basis for $\mathcal{A}(R_K, L)$, one may readily express the operator $\frac{D_{E'}^i}{i!}$ in terms of $\{\frac{D_E^i}{i!}\}$, and vice versa, by using the co-ordinate free definition (12).

Definition 11. Let $z \in R_L$ and let E be as above. Define $\mu_{E,z}$ to be the unique measure given by

$$\int_{R_K} G_{E,k}(x)\, d\mu_{E,z} = z^k\,,$$

for non-negative k. If $f(x) = \sum b_n G_{E,n}(x) \in \mathcal{C}(R_K, L)$, define

$$\hat{f}_E(z) := \int_{R_K} f(x)\, d\mu_{E,z}(x) = \sum b_n z^n\,. \tag{17}$$

The map $f \mapsto \hat{f}_E(z)$ gives a Banach space isomorphism between $\mathcal{C}(R_K, L)$ and the Tate-algebra $L\langle\langle z\rangle\rangle$ of power-series in z converging on the unit disc. With this isomorphism, the operator $\frac{D_E^i}{i!}$ transforms into the usual divided-derivative operator $\frac{D_z^i}{i!} = \frac{\frac{d^i}{dz^i}}{i!}$; i.e.,

$$\widehat{\frac{D_E^i}{i!} * f(x)} = \frac{D_z^i}{i!}\hat{f}(z)\,. \tag{18}$$

Finally, note that the divided-derivative $\frac{D_z^i}{i!}$ can also be obtained via digit expansions (as in Equation 14) using the Banach basis $\{z^{r^i}\}$ for the space of $\mathbb{F}_r$-linear elements inside $L\langle\langle z\rangle\rangle$.

2.3.2. *The p-adic case.* Let $R_K = \mathbb{Z}_p$ and $K = \mathbb{Q}_p$, etc. Using the Mahler basis $\{\binom{x}{i}\}$, it is very well-known that the convolution algebra of L-valued measures on $\mathbb{Z}_p$ is isomorphic to $\mathcal{R} := L \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[X]]$. Here a measure μ corresponds to the power series

$$F_\mu(X) = \sum_{k=0}^{\infty}\left(\int_{\mathbb{Z}_p}\binom{x}{k}\,d\mu(x)\right)X^k\,.$$

Clearly, then, we need only compute the action of the measure associated to X on $\mathcal{C}(\mathbb{Z}_p, L)$. Thus let $f(x) = \sum_{k=0}^{\infty} c_k\binom{x}{k}$ be a continuous function where $\{c_k\} \subset L$ and $c_k \to 0$ as $k \to \infty$. As $\binom{x+y}{n} = \sum_{j=0}^{n}\binom{x}{n-j}\binom{y}{j}$, one immediately computes

$$X * f(x) = \sum_{k=0}^{\infty} c_k\binom{x}{k-1} = \Delta f(x)\,, \tag{19}$$

where $\Delta f(x) := f(x+1) - f(x)$ is the usual difference operator.

Let $m \in L$ have $|m|_p < 1$ and let $f_m(x) := (1+m)^x$. Let μ be a $\mathbb{Q}_p$-valued measure on $\mathbb{Z}_p$ corresponding to a formal power series $F_\mu(X)$. One checks easily that

$$\mu * f_m(x) = F_\mu(m) f_m(x)\,, \tag{20}$$

so that the functions $f_m(x)$ are eigenfunctions for the operators $T_\mu\colon f \mapsto \mu * f$. It is simple to see that, up to scalars, these are all the common eigenfunctions for $\{T_\mu\}_{\mu\in\mathcal{M}(\mathbb{Z}_p,L)}$ defined over L.

Remark 4. In Part 4 of the Appendix to [Ko1], there is an exposition of the p-adic spectral theorem of Vishik. Let $\mathbb{C}_p$ be the completion of the algebraic closure of $\mathbb{Q}_p$ with the canonical extension of $|?|_p$. Vishik's theorem applies to "analytic operators;" i.e., operators A with compact spectrum σ_A over $\mathbb{C}_p$ and with analytic resolvent $R_A(z) := (z - A)^{-1}$ on the complement of σ_A. (Here analyticity means essentially that the matrix elements of the operator are Krasner analytic; for more see [Ko1].) Let μ be the measure on $\mathbb{Z}_p$ corresponding to X as above. One checks that the spectrum of the operator $f \mapsto \mu * f = \Delta f$ over $\mathbb{C}_p$ is given by the eigenvalues $\{x \in \mathbb{C}_p \mid |x|_p < 1\}$ which is obviously not a compact set. It is however, bounded and therefore it is reasonable to ask whether a form of Vishik's results might also hold for Δ.

One can also express the action of $L \otimes \mathbb{Z}_p[[X]]$ on $\mathcal{C}(\mathbb{Z}_p, L)$ via the following construction. Let B be the Banach space of bounded sequences $\{b_i\}_{i\in\mathbb{Z}} \subset L$ equipped with the sup norm. We write these sequences formally as $f(X, X^{-1}) = \sum_{i=-\infty}^{\infty} b_i X^i$.

Let H be the subspace of all $f(X, X^{-1}) = \sum b_i X^i \in B$ with $b_i \to 0$ as $i \to -\infty$ (note the minus sign!). In other words, H consists of all $f(X, X^{-1})$ whose polar part converges for

$X^{-1} \in R_L$. A little thought establishes that H is actually a closed $\mathcal{R}$-submodule of B where the action of $\mathcal{R} := L \otimes \mathbb{Z}_p[[X]]$ is via multiplication of power series in the usual sense. Thus $H/X\mathcal{R}$ is Banach space isomorphic to the Tate algebra $L\langle\langle X^{-1}\rangle\rangle$ and equips $L\langle\langle X^{-1}\rangle\rangle$ with a natural $\mathcal{R}$-module structure. Intuitively we simply multiply the two series in the "usual" fashion and throw out the terms X^i with i positive.

Let $X^{-1} \in R_L$ and let $f(x) = \sum_{k=0}^{\infty} c_k \binom{x}{k} \in \mathcal{C}(R_K, L)$. Let μ_X be the measure on $\mathbb{Z}_p$ given by

$$\int_{\mathbb{Z}_p} \binom{x}{k} \, d\mu_X(x) = X^{-k}\,,$$

for non-negative k. Set $\hat{f}(X) = \sum_{k=0}^{\infty} c_k X^{-k} = \int_{\mathbb{Z}_p} f(x)\, d\mu_X(x)$, as in Equation 17, which again gives a Banach space isomorphism between the continuous functions and the Tate algebra in X^{-1}. With this isomorphism, the action of the measure associated to $F(X) = \sum b_i X^i$ on f transforms into the action of $F(X) \in \mathcal{R}$ on $\hat{f}(X)$ presented above. Our thanks to W. Sinnott for pointing this construction out to us.

2.3.3. *A curious connection between continuity on $\mathbb{Z}_p$ and finite characteristic measures.* Let L be a finite extension of $\mathbb{Q}_p$ and let $f\colon \mathbb{Z}_p \to L$ be a continuous function written $\sum b_n \binom{x}{n}$. Set $a_n := f(n)$ for $n = 0, 1...$ and form two divided power series $A(x) := \sum a_n \frac{x^n}{n!}$ and $B(x) := \sum b_n \frac{x^n}{n!}$. It is well-known that $e^{-x}A(x) = B(x)$. Conversely, if one defines $B(x)$ by this equation for any sequence $\{a_n\}$, then we obtain a continuous function f with $f(n) = a_n$ if and only if b_n tends to 0 as $n \to \infty$.

Such a formalism also works when L has finite characteristic (see §8.4 of [Go4]). The results, and notation, of Subsection 2.3.1 then give the following curious result. Let μ be a measure on R_K and let $\{b_n\}$ be its measure coefficients with respect to $\{G_{E,n}(x)\}$. Let ν be the unique measure with coefficients $\{(-1)^n\}$ with respect to $\{G_{E,n}(x)\}$. Then *there is a continuous L-valued function $f\colon \mathbb{Z}_p \to L$ with $f(n) = b_n$, all n, if and only if the measure coefficients of $\nu * \mu$ tend to* 0.

The condition that the measure coefficients tend to 0 is stronger than what is needed to integrate continuous functions. Perhaps there is a larger class of functions that may be integrated by such a measure.

3. L-functions of τ-sheaves

3.1. **τ-sheaves.** The concept of a τ-sheaf ([TW1], [TW2], [BP1], [Boc1], [Ga1], [Ga3], [Go8]) arose out of the concept of a Drinfeld module which is where we begin. Let C be a smooth, projective, geometrically connected curve over the finite field $\mathbb{F}_r$ where $r = p^{m_0}$ with p prime. As usual one chooses a fixed closed point ∞ of degree (over $\mathbb{F}_r$) d_∞. The space $C' := C - \infty$ is affine and one denotes by A the ring of functions which are regular on C'. As is well known, the ring A is a Dedekind domain with unit group ${\mathbb{F}_r}^*$ and finite class group. Let k be the quotient field of A.

A field L with an $\mathbb{F}_r$-algebra map $\imath\colon A \to L$ is said to be an "A-field;" the kernel $\mathfrak{p}$ of $\imath$ is a prime ideal of A which is called the "characteristic of L." Let $\bar{L}$ be a fixed algebraic closure of L (which is obviously also an A-field with the same characteristic) and let $\tau\colon \bar{L} \to \bar{L}$ be the r-th power mapping, $\tau(l) = l^r$. The elements $l \in L$ and τ generate, by composition, an algebra of endomorphisms $L\{\tau\}$ of $\bar{L}$ with $\tau l = l^r \tau$, etc. There is unique homomorphism $D\colon L\{\tau\} \to L$ given by $D(\sum_{j=0}^{t} b_j \tau^j) = b_0$. A *Drinfeld A-module* ψ over L is an injection of

$\mathbb{F}_r$-algebras $\psi\colon A \to L\{\tau\}$, $a \mapsto \psi_a(\tau)$, such that $D \circ \psi = \imath$ but $\psi_a \neq \imath(a)\tau^0$ for some $a \in A$. We denote by $\psi[a]$ the finite subgroup of elements $z \in \bar{L}$ with $\psi_a(z) = 0$. As A is obviously commutative, $\psi[a]$ inherits an A-module structure. One can show the existence of an integer $d > 0$ such that $\psi[a]$ is A-module isomorphic to $A/(a)^d$ for all $a \in A - \mathfrak{p}$. One calls d *the rank of* ψ.

The next key step was taken by Drinfeld and then Anderson. Let $M := L\{\tau\}$ which we now view as an $L \otimes_{\mathbb{F}_r} A$-module in the following fashion. Let $f(\tau) = \sum_{i=0}^{j} c_i \tau^i$ be an arbitrary element of M, $a \in A$, and $l \in L$. One sets

$$l \otimes a \cdot f(\tau) := l f(\psi_a(\tau)) \,. \tag{21}$$

One checks that M then becomes a projective $L \otimes A$-module of rank d and thus gives rise to a locally-free sheaf on $\operatorname{Spec}(L \otimes A)$. See [An1], for example, for more on how the properties of ψ may be reinterpreted in terms of M.

The module M also possesses an obvious action by τ, with $(\tau, f(\tau)) \mapsto \tau f(\tau)$ (multiplication in $L\{\tau\}$). Note that for $l \in L$ one has

$$\tau l f(\tau) = l^r f(\tau) \,. \tag{22}$$

The essential features of M is that it is a coherent sheaf with "A-coefficients" equipped with a τ-action as above. This leads directly to the basic notion of a τ-sheaf. Thus let X be a scheme over $\mathbb{F}_r$. Let $\sigma = \sigma_X$ be the absolute Frobenius morphism with respect to $\mathbb{F}_r$; that is for any affine $\operatorname{Spec}(B) \subseteq X$ and $b \in B$ one has $\sigma^* b = b^r$.

Definition 12. A *τ-sheaf* on X is a pair $\underline{\mathcal{F}} := (\mathcal{F}, \tau)$ consisting of a locally-free sheaf $\mathcal{F}$ on $X \times_{\mathbb{F}_r} C'$ and an $(\mathcal{O}_X \otimes A)$-linear morphism

$$\tau = \tau_{\mathcal{F}} \colon (\sigma \times \mathrm{id})^* \mathcal{F} \to \mathcal{F} \,. \tag{23}$$

A *morphism of τ-sheaves* is a morphism of the underlying coherent sheaves which commutes with the τ-actions.

The reader will readily see that the Frobenius-linear property expressed in Equation 22 is equivalent to the formulation given in Equation 23.

Remark 5. In the papers [Boc1], [Boc2],[BP1] and [Go7] a more general notion of τ-sheaf is given where the underlying module need only be a coherent module.

3.2. **Domain spaces.** We will present here the basic ideas on exponentiation of ideals as in Section 8.2 of [Go4] and [Boc1].

3.2.1. *The theory at ∞.* We begin with the place ∞. Let $K := k_\infty$ now denote the completion of k at ∞ and let $\pi \in K$ now denote a fixed uniformizer of K; this is notation that henceforth will be used throughout the rest of this paper. Let $\mathbb{F}_\infty = \mathbb{F}_K \simeq \mathbb{F}_{r^{d_\infty}}$ be the associated finite field. Thus every element $\alpha \in K^*$ can be written uniquely as

$$\alpha = \zeta_\alpha \pi^{n_\alpha} \langle \alpha \rangle \,, \tag{24}$$

where $\zeta_\alpha \in \mathbb{F}_\infty^*$, $n_\alpha = v_\infty(\alpha) \in \mathbb{Z}$, and $\langle \alpha \rangle$ belongs to the group $U_1(K)$ of 1-units of K. Note that both ζ_α and $\langle \alpha \rangle$ depend on the choice of π. For $\alpha \in k^*$ we set

$$\deg_k(\alpha) = -d_\infty n_\alpha \,; \tag{25}$$

as usual we set $\deg_k(0) = -\infty$.

For any non-zero fractional ideal I of A, we also let $\deg_k(I)$ be the degree over $\mathbb{F}_r$ of the divisor associated to I on the curve C'; thus $\deg_k(I) = \deg_{\mathbb{F}_r} A/I$ when $I \subset A$. Moreover Equation 25 implies that for $\alpha \in k^*$ one has

$$\deg_k(\alpha) = \deg_k(\alpha A)\,. \tag{26}$$

Thus $\deg_k(\alpha)$ is the degree of the finite part of the divisor of α on the complete curve C.

We let $\mathbb{C}_\infty$ be the completion of a fixed algebraic closure $\bar{K}$ equipped with the canonical extension of the normalized absolute value $|?|_\infty$.

The element α is said to be *positive* (or *monic*) if and only if $\zeta_\alpha = 1$ (so that the notion of positivity most definitely depends on the choice of π). Clearly the product of two positive elements is also positive.

Let α be positive.

Definition 13. 1. We set $S_\infty := \mathbb{C}_\infty^* \times \mathbb{Z}_p$.
2. Let $s = (x, y) \in S_\infty$. We then set

$$\alpha^s := s^{\deg_k(\alpha)} \langle \alpha \rangle^y\,.$$

As $\langle \alpha \rangle = 1 + \lambda_\alpha$ with $v_\infty(\lambda_\alpha) > 0$, one has the convergent expression

$$\langle \alpha \rangle^y = \sum_{j=0}^{\infty} \binom{y}{j} \lambda_\alpha^j\,. \tag{27}$$

The space S_∞ will be the domain for the L-series of τ-sheaves at ∞.

The group action on S_∞ will be written additively. Suppose that $j \in \mathbb{Z}$ and α^j is defined in the usual sense of the canonical $\mathbb{Z}$-action on the multiplicative group. Let $\pi_* \in \mathbb{C}_\infty^*$ be a fixed d_∞-th root of π. Set $s_j := (\pi_*^{-j}, j) \in S_\infty$. One checks easily that Definition 13 gives $\alpha^{s_j} = \alpha^j$. When there is no chance of confusion, we denote s_j simply by "j."

Let $\mathcal{I}$ be the group of fractional ideals of the Dedekind domain A and let $\mathcal{P} \subseteq \mathcal{I}$ be the subgroup of principal ideals. Let $\mathcal{P}^+ \subseteq \mathcal{P}$ be the subgroup of principal ideals which have positive generators. One knows that $\mathcal{I}/\mathcal{P}^+$ is a finite abelian group. The association

$$\mathfrak{h} \in \mathcal{P}^+ \mapsto \langle \mathfrak{h} \rangle := \langle \lambda \rangle\,, \tag{28}$$

where λ is the unique positive generator of $\mathfrak{h}$, is obviously a homomorphism from $\mathcal{P}^+$ to $U_1(K)$.

For the moment, let $u = 1 + m \in U_1(\mathbb{C}_\infty)$, $|m| < 1$, be an arbitrary 1-unit in $\mathbb{C}_\infty$ and let $y = \sum_{j=j_0}^{\infty} c_j p^j$ be an arbitrary element of $\mathbb{Q}_p$. One sets $u^y := \prod_j (1 + m^{p^j})^{c_j}$ which obviously converges in $\mathbb{C}_\infty$. Thus $U_1(\mathbb{C}_\infty)$ is naturally a $\mathbb{Q}_p$-vector space; in particular, $U_1(\mathbb{C}_\infty)$ is thereby a divisible, and thus injective, group. We therefore have the next result.

Lemma 1. *The homomorphism* $\langle ? \rangle \colon \mathcal{P}^+ \to U_1(\mathbb{C}_\infty)$ *extends uniquely to a homomorphism* $\langle ? \rangle \colon \mathcal{I} \to U_1(\mathbb{C}_\infty)$.

The uniqueness in Lemma 1 follows from the fact that $\mathcal{P}^+$ has finite index in $\mathcal{I}$.

Definition 14. Let $I \in \mathcal{I}$ and $s = (x, y) \in S_\infty$. We then set

$$I^s := x^{\deg_k I} \langle I \rangle^y\,. \tag{29}$$

The reader will easily see that the mapping

$$\mathcal{I} \times S_\infty \mapsto I^s$$

is bilinear.

Definition 15. Let $\mathbb{V} \subset \mathbb{C}_\infty := k(\{I^{s_1} \mid I \in \mathcal{I}\})$. We call $\mathbb{V}$ the *value field* associated to π and π_*. The place on $\mathbb{V}$ given by its inclusion in $\mathbb{C}_\infty$ will be also be denoted ∞ and is called the *canonical infinite place of* $\mathbb{V}$.

Proposition 1. *The field* $\mathbb{V}$ *is a finite extension of* k.

Proof. If $I = (i)$ where i is positive, then $I^{s_1} = i \in k$. As $\mathcal{I}/\mathcal{P}^+$ is finite, the result follows. □

Let $\mathcal{O}_\mathbb{V} \subset \mathbb{V}$ be the ring of A-integers. The places of $\mathbb{V}$ which lie outside of $\operatorname{Spec}(\mathcal{O}_\mathbb{V})$ (and so lie over the place ∞ of k) are the "infinite primes of $\mathbb{V}$;" thus places lying above $\operatorname{Spec}(A)$ are the "finite primes."

Let α be an element of $\mathbb{V}$. We let $\deg_\mathbb{V}(\alpha)$ be the degree over $\mathbb{F}_r$ of the finite part of the divisor of α; as the degree of a principal divisor is 0, this is the opposite of the degree of the infinite part of the divisor of α. In particular, for $\alpha \in k$, one has

$$\deg_\mathbb{V}(\alpha) = [\mathbb{V} \colon k] \deg_k(\alpha)\,. \tag{30}$$

Similarly, if J is a fractional $\mathcal{O}_\mathbb{V}$-ideal, then we let $\deg_\mathbb{V}(J)$ be the degree over $\mathbb{F}_r$ of the associated finite divisor, etc. If I is an A-fractional ideal then we again have

$$\deg_\mathbb{V}(I\mathcal{O}_\mathbb{V}) = [\mathbb{V} \colon k] \deg_k(I)\,. \tag{31}$$

The next proposition and corollary, which are elementary, are explicitly recorded as they will be used in the proof of our main result.

Proposition 2. *Let* $0 \neq \alpha \in \mathcal{O}_\mathbb{V}$. *Then* $\deg_\mathbb{V}(\alpha) \geq 0$.

Corollary 1. *The only element of* $\mathcal{O}_\mathbb{V}$ *of negative degree is* 0.

Proposition 3. *Let* I *be a non-trivial ideal of* A. *Then,*

$$I\mathcal{O}_\mathbb{V} = I^{s_1}\mathcal{O}_\mathbb{V}\,. \tag{32}$$

Moreover, I^{s_1} *has a pole at every infinite place of* $\mathbb{V}$. *Finally, let* $\mathfrak{P}$ *be a prime of* $\mathcal{O}_\mathbb{V}$ *with additive valuation* $v_\mathfrak{P}(?)$. *Then*

$$\deg_k(\mathfrak{P}) v_\mathfrak{P}(I^{s_1}) \leq [\mathbb{V} \colon k] \deg_k(I)\,. \tag{33}$$

Proof. Let t be the order of $\mathcal{I}/\mathcal{P}^+$; thus $I^t = (\alpha)$ where $\alpha \in A$ is positive. By definition one has $\lambda^t = \alpha$ where $\lambda = I^{s_1}$. Thus the order of I^{s_1} is the same as the order of I at each prime of $\mathcal{O}_\mathbb{V}$, and Equation 32 follows. Clearly α has a pole at every infinite prime of $\mathbb{V}$ and so therefore must I^{s_1}. Finally, Equation 32 implies that $\deg_\mathbb{V}(I\mathcal{O}_\mathbb{V}) = \deg_\mathbb{V}(I^{s_1})$ as a principal divisor has degree 0. Thus Equation 33 follows from Equation 31. □

In particular, while the elements $\{I^{s_1}\}$ are not necessarily in A, they do behave very much like elements of A. For instance, they have the same absolute value at the (non-normalized) extension of $|?|_\infty$ at *any* infinite place of $\mathbb{V}$. This will be of importance in the proof of our main result.

3.2.2. *The theory for finite primes.* Let v be the place associated to a prime $\mathfrak{p}$ of A and set $d_v := \deg_{\mathbb{F}_r}(v)$. Let k_v be the associated completion of k with normalized absolute value $|?|_v$. Let $\bar{k}_v$ be a fixed algebraic closure of k_v and let $\mathbb{C}_v$ be its completion with the canonical extension of $|?|_v$. Finally, let $\sigma\colon \mathbb{V} \to \bar{k}_v$ be an embedding over k and set

$$k_\sigma = k_{\sigma,v} := k_v(\sigma(\mathbb{V}))\,. \tag{34}$$

By Proposition 1 one sees that $k_{\sigma,v}$ is finite over k_v and one lets $A_\sigma = A_{\sigma,v} \subset k_{\sigma,v}$ denote the ring of A_v-integers, with maximal ideal $M_{\sigma,v}$, and residue degree $f_\sigma = f_{\sigma,v}$. Any element $\beta \in A^*_{\sigma,v}$ can then be written

$$\beta = \omega_{\sigma,v}(\beta)\langle\beta\rangle_{\sigma,v}\,, \tag{35}$$

where $\omega_{\sigma,v}(\beta)$ belongs to the group $\mu_{r^{d_v f_\sigma}-1}$ of roots of unity inside $A_{\sigma,v}$, and where $\langle\beta\rangle_{\sigma,v}$ is a 1-unit.

Definition 16. We set

$$S_\sigma = S_{\sigma,v} := \varprojlim_j \mathbb{Z}/\left(p^j(r^{d_v f_\sigma}-1)\right) \simeq \mathbb{Z}_p \times \mathbb{Z}/(r^{d_v f_\sigma}-1)\,. \tag{36}$$

Let $y_\sigma = (y_{\sigma,0}, y_{\sigma,1}) \in S_\sigma$ and let β be as in Equation 35. Then we set

$$\beta^{y_\sigma} := \omega_{\sigma,v}(\beta)^{y_{\sigma,1}}\langle\beta\rangle_{\sigma,v}^{y_{\sigma,0}}\,. \tag{37}$$

Let $\mathcal{I}(v)$ be the group of A-fractional ideals generated by the primes $\neq v$ and let $I \in \mathcal{I}(v)$. One knows that $\sigma(I^{s_1}) \in A^*_{\sigma,v}$ by Equation 32. Let $s_\sigma = (x_v, y_\sigma) \in \mathbb{C}_v^* \times S_\sigma$. Finally we define

$$I^{s_\sigma} := x_v^{\deg_{\mathbb{F}_r}(I)}\sigma(I^{s_1})^{y_\sigma}\,. \tag{38}$$

The space $\mathbb{C}_v^* \times S_\sigma$ is the domain for the v-adic theory of L-series associated to τ-sheaves.

Note that if $I = (i)$ with i a positive v-adic unit, then

$$I^{s_\sigma} = x_v^{\deg_k(i)} i^{y_\sigma}\,.$$

In particular, one has $I^{(1,j,j)} = i^j$ for all integers j.

3.2.3. *Entire functions.* We begin here with the theory for S_∞. The theory for $\mathbb{C}_v^* \times S_\sigma$ is entirely similar and will be left to the reader. The basic reference is Section 8.5 of [Go4] (but see also [Boc1] in this regard).

Our L-series at ∞ will be "Dirichlet series" of the form $\sum c_I I^{-s}$ where $s = (x,y) \in S_\infty = \mathbb{C}_\infty^* \times \mathbb{Z}_p$. By Definition 14, $I^{-s} = x^{-\deg_k I}\langle I\rangle^{-y}$; therefore Dirichlet series immediately give rise to a 1-parameter family of formal power series in x^{-1}. This leads to the following definition.

Definition 17. An *entire function on* S_∞ is a $\mathbb{C}_\infty$-valued uniformly convergent family of entire power series in x^{-1} parameterized by $\mathbb{Z}_p$.

As explained in [Go4] this means that for each $y \in \mathbb{Z}_p$ we are given a power series $g_y(1/x)$ where $g_y(u)$ converges for all u. Moreover, for each bounded set $B \subset \mathbb{C}_\infty$ and $\epsilon > 0$ there exist a $\delta := \delta_B > 0$ such that if y_0 and y_1 are in $\mathbb{Z}_p$ with $|y_0 - y_1|_p < \delta$ then $|g_{y_0}(u) - g_{y_1}(u)|_\infty < \epsilon$ for all $u \in B$. This forces the zeroes to "flow continuously."

Let $0 \neq \rho \in \mathbb{C}_\infty$. Then, exactly as in Definition 6, one can define a norm on the space of functions analytic on the closed disc $\{u \in \mathbb{C}_\infty \mid |u|_\infty \leq t = |\rho|\}$. In [Go4], Definition 17 is reinterpreted in terms of the family of norms defined by all such ρ.

In [Boc1], one chooses a family $\{\rho_j\}_{j=0}^{\infty} \subset \mathbb{C}_\infty^*$ of increasing and unbounded norm to create a Fréchet space out of the entire power series in u. An entire function on S_∞, as in Definition 17, is then just a continuous function from $\mathbb{Z}_p$ into this Fréchet space.

As will be seen, the Dirichlet series of τ-sheaves are entire in the sense of Definition 17. Suppose for the moment that $A = \mathbb{F}_r[T]$ and let $L(s) = \sum_{a\in A \text{ positive}} c_a a^{-s}$, where $\{c_a\} \subset A$, be entire as above. Let j be a non-negative integer and set

$$z_L(x, -j) := L(\pi_*^j x, -j)\,. \tag{39}$$

It is easy to see that $z_L(x, -j)$ is now a power series in x^{-1} with $\mathbb{F}_r[T]$-coefficients. Moreover, as $L(s)$ is entire in s, we conclude that $z_L(x, -j)$ is entire in x^{-1}, and, in particular, its coefficients must tend to 0 in $\mathbb{F}_r((1/T))$. As the degrees of non-zero elements in A are obviously non-negative, we see that the only way this can happen is that almost all such coefficients vanish; i.e., $z_L(x, -j)$ must belong to $A[x^{-1}]$.

In fact, the rationality result just established for $\mathbb{F}_r[T]$ also holds for all L-series of τ-sheaves, for *any* A, and leads to the next definition. Let A now be completely general.

Definition 18. Let $L(s) = \sum_I c_I I^{-s}$ be an entire function where $\{c_I\} \subset \mathbb{C}_\infty$ lies in a finite extension H of k. Put $H_1 := H \cdot \mathbb{V}$ and define $z_L(x, -j)$ for $j \geq 0$ exactly as in Equation 39. We then say that $L(s)$ is *essentially algebraic* if and only if $z_L(x, -j) \in H_1[x^{-1}]$ for all j.

Let v be a finite prime of A. We leave to the reader the easy translation of "entire" and "essentially algebraic" to v-*adic Dirichlet series* of the form $\sum_{I \in \mathcal{I}(v)} c_I I^{-s_\sigma}$, $s_\sigma \in \mathbb{C}_v^* \times S_\sigma$ as in Equation 38.

The idea behind all of this is that one starts with a Dirichlet series $L(s) = \sum_I c_I I^{-s}$, $s \in S_\infty$, with coefficients in some finite extension H of k and then using the injections σ of H into $\bar{k}_v$, one defines the various "interpolations" $L(s_\sigma)$ of this L-series at the finite primes via the *same* sum *except* that we have removed the factors lying over v; i.e., $L(s_\sigma) := \sum_{I \in \mathcal{I}(v)} \sigma(c_I) I^{-s_\sigma}$. Therefore these interpolations are defined a priori simply as certain v-adic Dirichlet series. (See also Remark 6 just below.) In general there is no reason to expect any relationship between the interpolations of $L(s)$ at different places. However, the L-series of a τ-sheaf will turn out to be an essentially algebraic entire function. The *special polynomials* $\{z_L(x, -j)\}_{j=0}^{\infty}$ then have two basic attributes. First of all, by the work of Böckle and Pink[BP1], one can express them *cohomologically* (this is how one deduces in general that the power series $z_L(x, -j)$ is indeed a polynomial). Secondly, they allow one to relate the ∞-adic and v-adic theories by simply removing the finite number of Euler-factors lying above v in the special polynomials and then substituting x_v for x, etc. In other words, one obtains the functions $L(s)$ and $L(s_\sigma)$ associated to an essentially algebraic Dirichlet series by simply interpolating the special polynomials $\{z_L(x, -j)\}_{j=0}^{\infty}$; as the non-positive integers are *dense* in $\mathbb{Z}_p$ and S_v. This is obviously not possible more generally.

Remark 6. In the case where A is a principal ideal domain (e.g., $A = \mathbb{F}_r[T]$), the collection of interpolations given above ranges over *all* the places of the quotient field k. However, when A is general, there is no reason to believe that $\mathbb{V}$ has only one infinite place (corresponding to its inclusion in $\mathbb{C}_\infty$). Thus let $\sigma\colon \mathbb{V} \to \mathbb{C}_\infty$ be an injection corresponding to a different infinite place. Let I be an ideal of A and set

$$\langle I \rangle_\sigma := \pi_*^{\deg_k I} \sigma(I^{s_1})\,. \tag{40}$$

Using the fact that $\mathcal{I}/\mathcal{P}^+$ is finite, one concludes as before that $\langle I\rangle_\sigma$ is a unit (i.e., has norm 1) in $\mathbb{C}_\infty$. However, one cannot conclude that $\langle I\rangle_\sigma$ is a 1-unit; indeed there may be a non-trivial group of roots of unity one needs to handle. Nevertheless, these may be handled in exactly the same fashion as in the v-adic theory and one readily defines L-series associated to the infinite embedding σ. We will denote these by $L(\underline{\mathcal{F}}, s_{\sigma,\infty})$ where $s_{\sigma,\infty} = (x, y_{\sigma,\infty}) \in \mathbb{C}_\infty^* \times \varprojlim_j \mathbb{Z}/(r^{d_\sigma}-1)p^j$ where d_σ is the degree over $\mathbb{F}_r$ corresponding to the place of $\mathbb{V}$ given by σ. We will not stress these functions here as they have no classical counterparts and they may unnecessarily confuse the reader. However, when they arise from the L-series of a τ-sheaf (as presented in the next subsection), the reader may easily check that our techniques show that these functions also possess the same features as all other interpolations.

3.3. **Euler factors.** We now follow [BP1], [Boc1] and define the L-series of a τ-sheaf via an Euler product. Let X be a scheme of finite type over $\mathrm{Spec}(A)$ and let $\underline{\mathcal{F}}$ be a τ-sheaf as in Definition 12. Let X^0 be the set of closed points of X and for each $\alpha \in X^0$, let $\mathfrak{p}_\alpha$ be its image in $\mathrm{Spec}(A)$.

Definition 19. We define

$$L(\alpha, \underline{\mathcal{F}}, u)^{-1} := \det_k(I_d - u\tau \mid \mathcal{F}_\alpha \otimes_A k) \in k[u]\,, \tag{41}$$

where $\mathcal{F}_\alpha$ is the fiber of the sheaf $\mathcal{F}$, I_d is the identity morphism, and the determinant is taken over k.

In [BP1] it is shown that $L(\alpha, \underline{\mathcal{F}}, u)^{-1} \in A[u^{d_\alpha}]$ where d_α is the degree of α over $\mathbb{F}_r$. Let $d_{\mathfrak{p}_\alpha}$ be the degree of $\mathfrak{p}_\alpha$ over $\mathbb{F}_r$. Notice that $A[u^{d_\alpha}] \subseteq A[u^{d_{\mathfrak{p}_\alpha}}]$

Definition 20. Let $s \in S_\infty$. We set

$$L(\underline{\mathcal{F}}, s) := \prod_{\alpha \in X^0} L(\alpha, \underline{\mathcal{F}}, u)|_{u^{d_{\mathfrak{p}_\alpha}} = \mathfrak{p}_\alpha^{-s}}\,. \tag{42}$$

Böckle shows that the Euler product (42) converges on a "half-plane" of S_∞. That is, there exists a non-zero number $t \in \mathbb{R}$ such that (42) converges for all $s \in (x,y) \in S_\infty$ such that $|x|_\infty \geq t$.

Remark 7. For future use, we rewrite the condition for the half-plane in terms of additive valuations. Let K be the completion of k at ∞, as usual, and put $K_1 := K[\langle I\rangle]$ where I runs over the ideals of A; as $\mathcal{I}/\mathcal{P}^+$ is finite, K_1 is finite over K. (The reader will be tempted to conclude that K_1 is the compositum of K and $\mathbb{V}$. This is only obviously so when $d_\infty = 1$; the general relationship is not yet clear.) By abuse of notation let A_∞ denote the maximal compact subring of K and $A_{1,\infty}$ that of K_1. Let e be the ramification degree and v_∞ (resp. $v_{1,\infty}$) the canonical additive valuation on K (resp. K_1) which assigns 1 to a uniformizing parameter. Thus, upon extending these canonically to the algebraic closure, one has $v_{1,\infty} = ev_\infty$. Recall that we set $q_K = r^{\deg \infty}$. One then sees readily that

$$|x|_\infty \geq t \Leftrightarrow v_{1,\infty}(x) \leq -e\log_{q_K}(t)\,.$$

Let v now be a finite prime of A. The v-adic version of $L(\underline{\mathcal{F}}, s)$, which will be denoted $L(\underline{\mathcal{F}}, s_\sigma)$, $s_\sigma \in \mathbb{C}_v^* \times S_\sigma$, is now obvious using the embedding σ. Indeed, one simply uses in Definition 20, $\alpha \in X^0(v)$ where $X^0(v)$ is the set of closed points not lying over v. Also obvious in this case is the existence of a v-adic half-plane of convergence as the Euler factors all have coefficients in $\mathcal{O}_{\mathbb{V}}$.

Remark 8. In both the ∞-adic and v-adic theories, we abuse language and say that the Dirichlet series *converges absolutely* in the half-plane of convergence of its Euler product. Let $s \in S_\infty$, where $|x|_\infty \geq t$ (t as above), and expand $L(\underline{\mathcal{F}}, s)$ as $\sum_I b_I I^{-s}$. Then as $\deg I \to \infty$ one has $b_I I^{-s} \to 0$ which is what we shall mean by "absolute convergence." An exactly similar statement holds v-adically.

The next result is a restatement of one of the main theorems of [Boc1] and we refer the reader there for the proof.

Theorem 2. *Let X be a reduced, affine, equi-dimensional Cohen-Macaulay variety over Spec(A) of dimension e_X. Let $\underline{\mathcal{F}}$ be a τ-sheaf (which we recall is locally-free by definition in this paper). Then both $L(\underline{\mathcal{F}}, s)^{(-1)^{e_X - 1}}$ and $L(\underline{\mathcal{F}}, s_\sigma)^{(-1)^{e_X-1}}$ are essentially algebraic entire functions. Moreover, for each non-negative integer j, the degree in x^{-1} (respectively x_v^{-1}) of the associated special polynomial at $-j$ is $O(\log(j))$.*

Remarks 1. 1. In [Boc1] a slightly more restricted choice of local uniformizer π is chosen. This allows for certain global rationality statements that we have not given here. In any case, the arguments in [Boc1], and in particular, the growth estimate of the special polynomials, actually apply in complete generality as we have set things up.
2. The definition of the L-factors of τ-sheaves given in Definition 19 seems very different than the usual definition of, say, the L-factors of elliptic curves where one uses Tate modules and Frobenius morphisms etc. For τ-sheaves one can also use this approach and indeed one obtains the same local factors [Boc1]. For instance, the L-series of a Drinfeld module ψ has traditionally been defined this way. At the good primes, it is relatively easy to see that both approaches agree. At the bad primes, one needs to use the *Gardeyn maximal model* (e.g., [Ga1]) of the τ-sheaf associated ψ (which is analogous to the Néron model of an elliptic curve). In fact, the Euler factors of a Drinfeld module at the bad primes are remarkably similar to those of elliptic curves (ibid.). In particular, Theorem 2 gives the analytic continuation of the L-series of Drinfeld modules and general A-modules [An1], [Boc1] defined over finite extensions of k.

3.4. **The canonical 1-parameter family of measures associated to a τ-sheaf.** Let X, $\underline{\mathcal{F}}$, etc., be as in Theorem 2. Let $s = (x, y) \in S_\infty$ and set

$$L(s) = L(x, y) := L(\underline{\mathcal{F}}, s)^{(-1)^{e_X - 1}} = \sum_I b_I I^{-s}\,, \tag{43}$$

where I runs over the ideals of A. We know that $L(s)$ has the following properties:

(1) The coefficients $\{b_I\}$ belong to A.
(2) $L(s)$ converges absolutely in some half plane $\{(x, y) \mid |x|_\infty \geq t\}$ of S_∞.
(3) For each non-negative integer j, the power series $L(\pi_*^j x, -j)$ is a polynomial in x^{-1} (with coefficients in $\mathcal{O}_{\mathbb{V}}$) whose degree is $O(\log(j))$.

Definition 21. Any Dirichlet series $L(s)$ satisfying the above three properties will be said to be in the *motivic class* $\mathfrak{M}$

As will be seen by our main result Theorem 3, every *partial* L-series (in the sense of Definition 25 below) associated to a Dirichlet series in class $\mathfrak{M}$ will *also* be in $\mathfrak{M}$. Note, in particular, that Definition 21 definitely does *not* require $L(s)$ to have an associated Euler-product.

Remark 9. We use the adjective "motivic" in Definition 21 precisely because it is our expectation that the only general procedure to produce non-trivial Dirichlet series in class $\mathfrak{M}$ will be via partial L-series of τ-sheaves.

From now on we let $L(s) = \sum b_I I^{-s}$ be a fixed Dirichlet series in class $\mathfrak{M}$. For $\alpha \in A_{1,\infty}$ let δ_α be the Dirac measure concentrated at α as in Definition 4.

Definition 22. Let $x \in \mathbb{C}_\infty$ have $|x|_\infty \geq t$ (where t gives a half-plane of absolute convergence as above). Then we define

$$\mu_{L,x} := \sum_I b_I x^{-\deg_k I} \delta_{\langle I \rangle} \,, \tag{44}$$

where I runs over all ideals of A. We call $\mu_{L,x}$ *the canonical (1-parameter) family of measures* associated to $L(s)$ at ∞.

It is clear that, with x chosen as above, the series for $\mu_{L,x}$ converges to a bounded measure on $A_{1,\infty}$. As such, its coefficients with respect to any basis must also be bounded.

Let v be a prime of A and let $k_{\sigma,v}$ be as in Equation 34 with maximal compact subring $A_{\sigma,v}$. The v-adic version of Definition 22 is given next.

Definition 23. Let $x_v \in \mathbb{C}_v$ with $|x_v|_v \geq t_v > 1$. We define

$$\mu_{L,x_v} := \sum_I b_I x_v^{-\deg I} \delta_{\sigma(I^{s_1})} \,, \tag{45}$$

where, again, I ranges over *all* ideals of A. We call $\mu_{L,x,v}$ the *canonical family of measures associated to $L(s)$ at v.*

It is clear that μ_{L,x_v} also converges to a bounded measure with the above choice of x_v.

The reader should note, of course, that if $v \mid I$, then $\sigma(I^{s_1}) \notin A^*_{\sigma,v}$.

4. The main theorem

Let $L(s) = \sum b_I I^{-s}$ continue to be a Dirichlet series in class $\mathfrak{M}$.

4.1. **Partial L-series.** Let $W = \{w_1, \ldots, w_k\}$ be a finite collection of places of $\mathbb{V}$. We explicitly allow at most one element of W to be the canonical infinite prime of $\mathbb{V}$ and the rest are assumed to be finite places. (As in Remark 6, one may also use *all* the infinite primes of $\mathbb{V}$, and we leave to the reader the easy modifications necessary to include them.) Let $E = \{n_1, \ldots, n_k\}$ be a collection of positive integers and let $\mathfrak{w} = \mathfrak{w}_{W,E}$ be the effective divisor $\sum n_i w_i$ on C. Write $\mathfrak{w} = \mathfrak{w}_f + \mathfrak{w}_\infty$ where $\mathfrak{w}_f$ consists of the sum over the finite primes and $\mathfrak{w}_\infty$ is a multiple of ∞ (which may be 0 if $\infty \notin W$).

For each finite place w_i of W, let $\mathcal{O}_{w_i} = \mathcal{O}_{\mathbb{V},w_i}$ be the associated local ring. If $w_i = \infty$ we *define* $\mathcal{O}_\infty = A_{1,\infty}$. Let $\mathfrak{a} = \{\alpha_i \in \mathcal{O}_{w_i}\}$ be a collection of elements in these local rings.

Definition 24. Let I be an ideal of A. We say that $I \equiv \mathfrak{a} \pmod{\mathfrak{w}}$ if $I^{s_1} \equiv \alpha_i \pmod{w_i^{n_i}}$ for all finite w_i in $\mathfrak{w}$ and $\langle I \rangle \equiv \alpha_j \pmod{w_j^{n_j}}$ when w_j is the canonical infinite prime ∞.

Clearly, should the reader desire, one can use the approximation theorem to replace $\mathfrak{a}$ by an element of $\mathbb{V}$. Note also that if $w_j = \infty$ then we may assume that α_j is a 1-unit as otherwise the definition is vacuous.

Definition 25. Let $L(s) = \sum_I b_I I^{-s}$ be a Dirichlet series and let $\mathfrak{a}$, $\mathfrak{w}$ be as above. We set

$$L_{\mathfrak{a},\mathfrak{w}}(s) := \sum_{I \equiv \mathfrak{a} \pmod{\mathfrak{w}}} b_I I^{-s} \,. \tag{46}$$

The Dirichlet series $L_{\mathfrak{a},\mathfrak{w}}(s)$ is called the *partial Dirichlet series* associated to $L(s)$ at $\mathfrak{a}, \mathfrak{w}$.

The next two statements are then the main results of this paper.

Theorem 3. *Let $L(s)$ be a Dirichlet series in the motivic class $\mathfrak{M}$. Then $L(s)$ analytically continues to an essentially-algebraic entire function on S_∞. Moreover, any v-adic interpolation of $L(s)$, for $v \in Spec(A)$, analytically continues to an essentially-algebraic entire function on $\mathbb{C}_v^* \times S_\sigma$.*

Proof. We first show that $L(s)$ may be analytically continued to essentially algebraic entire functions at the canonical place ∞ and the finite primes of $\mathcal{O}_{\mathbb{V}}$. Our proof will be to express $L(s)$ as a uniform limit of entire functions. We then refine this result to establish to establish the last part of the result.

Let $\mu_{L,x}$ be the canonical family of measures associated to $L(s)$ as given by Equation 44. Let t be chosen so that $L(s)$ converges absolutely for $|x| \geq t$; thus for such x, the series for $\mu_{L,x}$ converges to a bounded measure. Let t_1 be a positive real number less than t. The proof proceeds by analytically extending $L(s)$, $s = (x, y) \in S_\infty$, from the half-plane of absolute convergence to $\{(x, y) \in S_\infty \mid |x|_\infty \geq t_1\}$ in a manner which is uniform in y and x. As t_1 is arbitrary the full analytic continuation follows.

Recall that K_1 is the extension of K obtained by adjoining $\langle I \rangle$ for all ideals I of A with maximal compact subring $A_{1,\infty}$ and maximal ideal $M_{1,\infty}$. Let $y \in \mathbb{Z}_p$ and let $z \in A_{1,\infty}$. Define

$$\tilde{z}^y := \begin{cases} z^y, & \text{for } z \equiv 1 \pmod{M_{1,\infty}} \\ 0, & \text{otherwise}\,, \end{cases} \tag{47}$$

where $z^y = (1 + (z - 1))^y$ is computed as before. Clearly the function $z \mapsto \tilde{z}^y$ is locally-analytic on $A_{1,\infty}$ of order 1. Moreover, one checks easily that $\|\tilde{z}^y\|_1 = 1$ for all y so that, as in Remark 2, we can obtain estimates on its expansion coefficients which are uniform in y.

For $|x|_\infty \geq t$ we have the basic integral

$$L(x, y) = \int_{A_{1,\infty}} \tilde{z}^{-y} \, d\mu_{L,x}(z) \,. \tag{48}$$

The analytic continuation of $L(s)$ proceeds by showing that this integral extends to $|x|_\infty \geq t_1$ for all t_1 as above.

Let $\{Q_n(z)\}_{n=0}^\infty$ be an orthonormal basis for the $\mathcal{C}(A_{1,\infty}, K_{1,\infty})$ consisting of polynomials of degree n, as in Subsection 2.1, with measure coefficients

$$\left\{ b_n = b_n(x) = \int_{A_1} Q_n(z) \, d\mu_{L,x}(z) \right\} \tag{49}$$

(N.B., the measure coefficients are now functions of the parameter x in $\mu_{L,x}$). For $|x| \geq t$, $b_n(x)$ is bounded as $\mu_{L,x}$ is bounded.

The next step is to show that $b_n(x)$ is a polynomial in x^{-1} whose degree is $O(\log(n))$. To see this write

$$Q_n(z) = \sum_{j=0}^n q_{n,j} z^j \,. \tag{50}$$

Thus

$$b_n(x) = \sum_{j=0}^{n} q_{n,j} \int_{A_{1,\infty}} z^j \, d\mu_{L,x}(z)\,. \tag{51}$$

By definition, $L(x, -j) = \int_{A_{1,\infty}} z^j \, d\mu_{L,x}(z)$ and, *as $L(s)$ is assumed to be motivic*, it is a polynomial in x^{-1} (with coefficients in K_1) whose degree is $O(\log(j))$. Therefore $b_n(x)$ is also a polynomial whose degree (in x^{-1}) d_n is $O(\log(n))$. We write

$$b_n(x) = \sum_{i=0}^{d_n} b_{n,i} x^{-i}\,. \tag{52}$$

In order to apply Theorem 1, we now switch from using the absolute value $|?|_\infty$ to the equivalent additive functions as in Remark 7. Let v_∞, $v_{1,\infty}$, e, q_K be as presented there. Then we see that $L(s)$ converges absolutely for all $\{(x, y) \in S_\infty \mid v_{1,\infty}(x) \leq -e\log_{q_K}(t)\}$. We know that $\mu_{L,x}$ converges to a bounded measure in this region. Moreover, by choosing t a bit larger, we can assume that $v_{1,\infty}(\mu_{L,x}(U)) \geq 0$ for any compact open U. As the integral of $Q_n(z)$ against such a measure must also satisfy the *same* bounds, we have

$$v_{1,\infty}(b_{n,i}) + ie\log_{q_K}(t) \geq 0 \tag{53}$$

uniformly in n. In other words, there is a non-negative constant C_1 such that

$$v_{1,\infty}(b_{n,i}) \geq -iC_1$$

uniformly in n.

Therefore for any t_1 sufficiently small we conclude the existence of a positive constant C_2 such that for $|x|_\infty \geq t_1$ we have $v_{1,\infty}(b_n(x)) \geq -d_nC_2$.

Now expand $\tilde{z}^y = \sum_n a_{y,n}Q_n(z)$ so that the basic integral (48) becomes

$$L(x, y) = \int_{A_{1,\infty}} \tilde{z}^{-y} \, d\mu_{L,x}(z) = \sum_{n=0}^{\infty} a_{-y,n} b_n(x)\,. \tag{54}$$

By Theorem 1 and Remark 2, we conclude that $v_{1,\infty}(a_{-y,n}) \geq C_3n$ uniformly in y for some positive constant C_3. Therefore $v_{1,\infty}(a_{-y,n}b_n(x)) \geq C_3n - C_2d_n$. As $d_n = O(\log(n))$, this goes to ∞ with n. In particular the series for the integral converges uniformly in x, for $|x|_\infty \geq t_1$, and uniformly in y thus giving the desired analytic continuation.

The analogous v-result is even easier since I^{s_1} is always a v-adic integer. $\square$

Theorem 4. *Let $L(s)$ be a Dirichlet series in the motivic class $\mathfrak{M}$. Then all partial Dirichlet series associated to $L(s)$ are also in the class $\mathfrak{M}$.*

Proof. We continue with the notations, etc., as in the proof of Theorem 3. Let $\mathfrak{a}, \mathfrak{w}$ and $L_{\mathfrak{a},\mathfrak{w}}(s)$ be as in Definition 25. The first two conditions for $L_{\mathfrak{a},\mathfrak{w}}(s)$ to be in the motivic class, as given in Subsection 3.4, are easily checked to follow directly from those of $L(s)$. Thus we only need check the more subtle third condition. Let j be a non-negative integer and

$$z_{L_{\mathfrak{a},\mathfrak{w}}}(x, -j) := L_{\mathfrak{a},\mathfrak{w}}(\pi_*^j x, -j)\,. \tag{55}$$

We need to show that $z_{L_{\mathfrak{a},\mathfrak{w}}}(x, -j)$ is a polynomial in x^{-1} whose degree grows like $O(\log(j))$.

To do this we note first that it suffices to handle the case where $\mathfrak{w}$ is supported on one place. Indeed, we simply use the result inductively at each place dividing $\mathfrak{w}$.

We begin by writing

$$z_{L_{\mathfrak{a},\mathfrak{w}}}(x,-j)=\sum_{i=0}^{\infty} c_{j,i}x^{-i}\,. \tag{56}$$

By hypothesis we know that $\{c_{j,i}\}\subseteq \mathcal{O}_{\mathbb{V}}$. Thus to show $c_{j,i}$ vanishes for some choice of j and i, by Corollary 1 we need only show that it has negative degree. We assume first that $\mathfrak{w}$ is supported at ∞; the v-adic case will follow in a similar fashion.

Thus suppose that $\mathfrak{w}=n_\infty\infty$ for $n_\infty\geq 1$ and $\mathfrak{a}=\alpha$ where α is a 1-unit in $A_{1,\infty}$. Let χ be the characteristic function of the open subset $\alpha+M_{1,\infty}^{n_\infty}$ of $A_{1,\infty}$. Obviously, χ is locally analytic of order $h=n_\infty$.

We then obtain the following integral representation for the partial L-series $L_{\mathfrak{a},\mathfrak{w}}(s)$

$$L_{\mathfrak{a},\mathfrak{w}}(s)=\int_{A_{1,\infty}}\chi(z)\tilde{z}^{-y}\,d\mu_{L,x}(z)\,. \tag{57}$$

Note that the norm of the locally analytic function of order n_∞, $\chi(z)\tilde{z}^{-y}$, is again 1 and obviously independent of y. Thus if we write $\chi(z)\tilde{z}^{y}=\sum_n f_{y,n}Q_n(z)$ we find

$$L_{\mathfrak{a},\mathfrak{w}}(s)=\sum_n f_{-y,n}b_n(x)\,, \tag{58}$$

where we have uniform estimates on $v_{1,\infty}(f_{-y,n})$ by Theorem 1; i.e., there is a positive constant C_4, independent of y (and $-y$), such that

$$v_{1,\infty}(f_{-y,n})\geq C_4 n\,. \tag{59}$$

Combining Equations 58 and 52 we obtain the explicit formula

$$L_{\mathfrak{a},\mathfrak{w}}(s)=L_{\mathfrak{a},\mathfrak{w}}(x,y)=\sum_{i=0}^{\infty}\left(\sum_{n=0}^{\infty} f_{-y,n}b_{n,i}\right)x^{-i}\,. \tag{60}$$

Thus with j as above we obtain

$$c_{j,i}=\pi_*^{-ji}\sum_{n=0}^{\infty} f_{j,n}b_{n,i}\,. \tag{61}$$

We now estimate the valuation of $\sum_{n=0}^{\infty} f_{-y,n}b_{n,i}$; using this estimate in Equation 61 will allow us to finish the proof. As the degree d_n of $b_n(x)$ is $O(\log(n))$ we conclude that if x^{-i} occurs with a non-zero coefficient in $L_{\mathfrak{a},\mathfrak{w}}(x,y)$ then x^{-i} must be contributed by those $b_n(x)$ where n is *exponential* in i. Combining this with with (59) we obtain the *fundamental estimate* (for i sufficiently large)

$$v_{1,\infty}\left(\sum_{n=0}^{\infty} f_{-y,n}b_{n,i}\right)\geq C_5e^{C_6 i}\,, \tag{62}$$

for positive C_5 and C_6; this is again independent of y. (The reader should note that $v_{1,\infty}(b_{n,i})$ is greater then a constant times i and so can be absorbed into the exponential term as given above.)

Let $b_{\mathbb{V}}$ be the number of infinite places of $\mathbb{V}$. Using the estimate (62) in Equation 61, we see that the contribution to $\deg_{\mathbb{V}}(c_{j,i})$ at the place ∞ of $\mathbb{V}$ must be $\leq[\mathbb{V}\colon k](-C_5e^{C_6 i}+ij)$. On the other hand, as $L_{\mathfrak{a},\mathfrak{w}}(s)$ has a half-plane of absolute convergence and coefficients in A (as mentioned above), the discussion in Subsection 3.2.1 assures us that the other $b_{\mathbb{V}}-1$

infinite places of $\mathbb{V}$ can contribute at most a positive constant times ij to the degree. We conclude that for sufficiently large i

$$\deg_{\mathbb{V}}(c_{j,i}) \leq C_7 ij - C_5 e^{C_6 i}\,, \tag{63}$$

for some positive constant C_7. Elementary estimates show that this expression is then negative for $i \gg \log(j)$ which gives the result.

Again the v-adic version follows similarly using the fact that the degree of a principal divisor on a complete curve must vanish. Finally, we note that the fundamental estimate (62) can also be used to give another proof of Theorem 3. □

Corollary 2. *Let $L(s)$ be as in* Theorem 3. *Let U be a compact open subset of $A_{1,\infty}$ (resp. $A_{\sigma,v}$). Then $\mu_{L,x}(U)$ (resp. $\mu_{L,x_v}(U)$) is a polynomial in x^{-1} (resp. x_v^{-1}).*

Proof. We can suppose U is of the form $\alpha + M^j_{1,\infty}$ (resp. $\alpha + M^j_{\sigma,v}$). Then $\mu_{L,x}(U)$ (resp. $\mu_{L,x_v}(U)$) is the value of the associated partial L-series at $y = 0$ (resp. $y_\sigma = 0$). Thus the result follows immediately from Theorem 3. □

The above proof of the analytic continuation depends crucially on the fact that $z \mapsto \tilde{z}^y$ is locally analytic. If one had *any* other type of locally analytic endomorphism of the group of 1-units, then the proof would *automatically* work for it also. Therefore the following question is of great interest.

Question 1. Let $K = \mathbb{F}_r((1/T))$ and let U_1 be the group of 1-units of K. Does there exist a locally-analytic endomorphism $f\colon U_1 \to U_1$ which is not of the form $u \mapsto u^y$ for some p-adic integer y?

It is reasonable to expect a negative answer to Question 1 but we certainly have no proof of this as of this writing.

5. Complements

Let k be as before and let $\mathcal{E}$ be an A-module [An1] [Boc1] defined over a finite extension k_1 of k. For each finite prime v of A we can define the v-adic Tate module of $\mathcal{E}$. Using the invariants of inertia and the Frobenius element at another finite prime $\mathfrak{p}$ not dividing v one obtains the local Euler-factors of the L-series of E at $\mathfrak{p}$ (as mentioned above). One can also use the same definitions when $\mathfrak{p}$ is an *infinite* place. Remarkably, as in [Ga1], one still obtains polynomials with coefficients in A and which are independent of v. For instance, in the case that $\mathcal{E}$ is a Drinfeld module (or just uniformizable) one finds that the coefficients are actually in $\mathbb{F}_r$ by using the associated lattice (in fact, by Gardeyn [Ga2] this essentially characterizes uniformizable A-modules).

As mentioned in [Boc1] (also [Go6]) these factors at ∞ should give rise to trivial-zeroes for the special polynomials of $L(\mathcal{E}, s)$, and thus also $L(\mathcal{E}, s)$ itself. In fact, one expects this to ultimately be a completely general phenomenon for all τ-sheaves. The trivial zeroes for the v-adic functions will be given by the Euler factors in the special polynomials lying over v (which are removed when one interpolates v-adically). In the case where $\mathcal{E}$ is uniformizable the trivial zeroes resemble the classical case. However, when $\mathcal{E}$ is arbitrary they represent something new. As of this writing these trivial zeroes have an extremely mysterious effect on the collection of all zeroes. Indeed, because of the non-Archimedean nature of S_∞ and the uniform continuity of $L(s)$, trivial zeroes influence nearby zeroes (called "near-trivial zeroes") as in [Go6]. One therefore wants to find the set of zeroes (called "critical zeroes")

which are not so influenced. This seems very hard at the moment. Classical theory suggests that it may require deeper understanding of the connections with modular forms as in [Boc2] (see also [Go7]).

Remark 10. We finish by explaining how elementary estimates can be used to imply logarithmic growth in general in the rank 1 case (as is evident in the explicit calculation given in [Th1]). Let $\mathcal{K}$ be any field of characteristic p and let $W \subseteq \mathcal{K}$ be a finite additive subgroup of order p^{m_W}. Define $s_i(W) := \sum_{w\in W} w^i$ for non-negative integers i. Then one has

$$0 \le i < p^{m_W} - 1 \Rightarrow s_i(W) = 0\,. \tag{64}$$

The proof, à la Carlitz, goes as follows. Put $e_W(z) := \prod_{w\in W}(z-w)$ which, by standard arguments, is an additive polynomial with constant derivative $\lambda_W \neq 0$. Logarithmic differentiation then implies

$$\lambda_W/e_W(z) = \sum_{w\in W}(z-w)^{-1} = \sum_{w\in W}\frac{1}{z}(1-w/z)^{-1}\,.$$

Using the geometric series, one finds that the coefficient of z^{-j} is $s_{j-1}(W)$. On the other hand, $\lambda_W/e_W(z)$ has a zero of order p^{m_W} at ∞ and the result follows. Applying this to the sums over finite subgroups which arise in the rank 1 case immediately gives the logarithmic growth of special polynomials.

Note that Equation 64 includes the very well-known result $\displaystyle\sum_{a\in\mathbb{F}_r} a^i = 0$ for $0 \le i < r-1$.

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